

# The multiformity of Lifshits tails caused by random Landau Hamiltonians with repulsive impurity potentials of different decay at infinity

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**ABSTRACT.** For a charged quantum particle in the Euclidean plane subject to a perpendicular constant magnetic field and repulsive impurities, randomly distributed according to Poisson's law, we determine the leading low-energy fall-off of the integrated density of states in case the single-impurity potential has either super-Gaussian or regular sub-Gaussian long-distance decay. The forms of the resulting so-called magnetic Lifshits tails reflect the great variety of these decays. On the whole, we summarize, unify, and generalize results in previous works of K. Broderix, L. Erdős, D. Hundertmark, W. Kirsch, and ourselves.

## 1. Introduction

The energy spectrum of a spinless quantum particle with mass  $m > 0$  and electric charge  $Q \neq 0$  in the Euclidean plane  $\mathbb{R}^2$  subject to a perpendicular constant magnetic field of strength  $B > 0$  is explicitly known since the early works of Fock [Foc28] and Landau [Lan30]. It consists only of isolated harmonic-oscillator like eigenvalues  $\varepsilon_0, 3\varepsilon_0, 5\varepsilon_0, \dots$  of infinite degeneracy, where  $\varepsilon_0 := \hbar |Q|B/2m$  is the so-called *lowest Landau level* and  $2\pi\hbar > 0$  is Planck's constant. The underlying "magnetic" Schrödinger operator

$$(1.1) \quad H(0) := \frac{1}{2m} \left[ \left( i\hbar \frac{\partial}{\partial x_1} - \frac{QB}{2}x_2 \right)^2 + \left( i\hbar \frac{\partial}{\partial x_2} + \frac{QB}{2}x_1 \right)^2 \right],$$

acting on the Hilbert-space  $L^2(\mathbb{R}^2)$  of Lebesgue square-integrable, complex-valued functions on  $\mathbb{R}^2$ , is often referred to as the *Landau Hamiltonian*. Here  $i = \sqrt{-1}$  stands for the imaginary unit and  $(x_1, x_2)$  for the pair of Cartesian co-ordinates of a given point  $x \in \mathbb{R}^2$  interpreted as the classical position of the particle. The spectral resolution of  $H(0)$  may be written as

$$(1.2) \quad H(0) = \varepsilon_0 \sum_{n=0}^{\infty} (2n+1)P_n.$$

The orthogonal projection  $P_n$  associated with the  $n$ th *Landau level*  $(2n+1)\varepsilon_0$  is an integral operator with a continuous kernel. Its diagonal  $P_n(x, x) = 1/(2\pi\ell^2)$ , given

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in terms of the so-called *magnetic length*  $\ell := \sqrt{\hbar/|Q|B}$ , is naturally interpreted as the degeneracy of the  $n$ th Landau level per area.

In recent decades, the fabrication of low-dimensional semiconductor microstructures and micro-devices as well as the discovery of the (integer) quantum Hall effect have stimulated investigations of the so-called *random Landau Hamiltonian*

$$(1.3) \quad H(V_\omega) := H(0) + V_\omega,$$

where  $V_\omega$  is some random potential modelling the interaction of the particle with irregularly distributed impurities. An important issue is to understand the spectral properties of the perturbed operator (1.3).

In this paper we choose  $V_\omega$  to be a repulsive Poissonian potential

$$(1.4) \quad V_\omega(x) := \sum_j U(x - q(\omega, j)), \quad U \geq 0.$$

Here for a given realization  $\omega \in \Omega$  of the randomness the point  $q(\omega, j) \in \mathbb{R}^2$  stands for the position of the  $j$ th impurity repelling the particle at  $x \in \mathbb{R}^2$  by a positive potential  $U$  which neither depends on  $\omega$  nor on  $j$ . We assume that the single-impurity potential  $U$  is strictly positive on some non-empty open set in  $\mathbb{R}^2$ . The impurities are supposed to be distributed “completely at random” on the plane. More precisely, the probability of simultaneously finding  $M_1, M_2, \dots, M_K$  impurity points in respective pairwise disjoint (Borel) subsets  $\Lambda_1, \Lambda_2, \dots, \Lambda_K \subset \mathbb{R}^2$  is given by the product  $\prod_{k=1}^K e^{-\varrho|\Lambda_k|} (\varrho|\Lambda_k|)^{M_k} / M_k!$ , where  $|\Lambda_k| := \int_{\Lambda_k} d^2x$  is the area of  $\Lambda_k$  and the parameter  $\varrho > 0$  is the mean concentration of impurities.

The simplest but physically important spectral characteristics of the random Landau Hamiltonian (1.3) is its *integrated density of states*  $N : E \mapsto N(E)$ . Roughly speaking,  $N(E)$  is the averaged number of energy levels per area below a given energy  $E \in \mathbb{R}$ . Under rather weak assumptions on  $U$  it can be shown that both the spectrum of  $H(V_\omega)$  and the set of growth points of  $N$  coincide for almost all  $\omega \in \Omega$  with the half-line  $[\varepsilon_0, \infty[$ .

For the unperturbed Landau Hamiltonian  $H(0)$  the graph of  $N$  looks like a staircase, where the distance of successive steps as well as their height are proportional to  $B$ , confer Figure 1. The presence of repelling impurities will lift the degeneracy of the Landau levels and reduce the values of  $N$ . Moreover, since the impurities are randomly distributed, the steps are expected to be smeared out the more the stronger the randomness is. For a sketch of the resulting graph of  $N$  for a given  $U$  and two different values of  $\varrho$  see again Figure 1. But the reader should notice that nobody really knows what  $N$  is looking like for  $\varrho > 0$ . In particular, there is not even a proof that  $N$  will become continuous for sufficiently strong randomness.

To our knowledge, the only rigorous results available so far for  $N$  of the random Landau Hamiltonian (1.3) with a Poissonian potential (1.4) concern its asymptotic high-energy growth [Mat93, Uek94] and low-energy fall-off [BHKL95, Erd98, HLW99]. The first asymptotics is neither affected by the impurities nor the magnetic field and given by

$$(1.5) \quad N(E) \sim \frac{m}{2\pi\hbar^2} E = \frac{(1/2\pi\ell^2)}{2\varepsilon_0} E \quad (E \rightarrow \infty),$$

which is consistent with a famous result of Weyl [Wey12]. The asymptotic behaviour of  $N$  near the bottom of the almost-sure spectrum  $[\varepsilon_0, \infty[$ , that is, the behaviour of

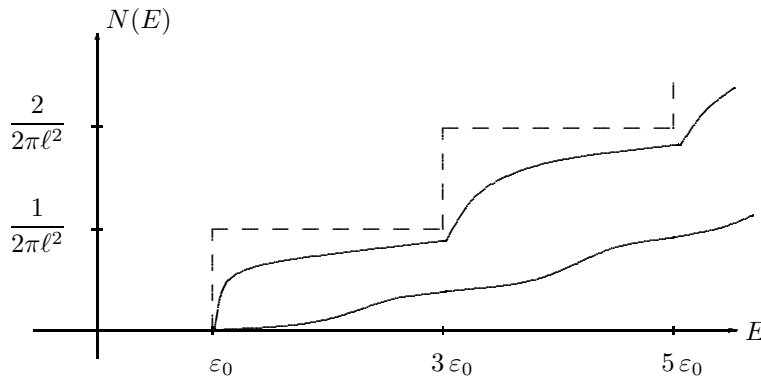


FIGURE 1. Qualitative plots of the integrated density of states  $N(E)$  of the random Landau Hamiltonian as a function of the energy  $E$ . The dashed line corresponds to the case without impurities ( $\varrho = 0$ ). The upper and lower solid line correspond to the presence of repulsive impurities with mean concentration  $\varrho_1 > 0$  and  $\varrho_2 > \varrho_1$ , respectively. By definition, the Lifshits tail is the leading asymptotic behaviour of  $N(E)$  as  $E$  approaches the lowest Landau level  $\varepsilon_0$  from above.

$N(\varepsilon_0 + E)$  for  $E \downarrow 0$  is more complicated and harder to obtain. Since the presence of the impurities should rarely especially low-lying energy levels, one expects that the values of  $N$  near  $\varepsilon_0$  are dramatically diminished such that  $N$  drops down to zero continuously. The resulting leading low-energy fall-off is commonly referred to as a *Lifshits tail* in honour of the theoretical physicist I. M. Lifshits, who was the first to develop a quantitative theory in the case  $B = 0$  [Lif64, LGP88]. The form of the Lifshits tail mainly depends on the decay of the impurity potential  $U(x)$  for long (Euclidean) distance  $|x| := \sqrt{x_1^2 + x_2^2}$ . This stems from the fact that low-lying energy levels mainly originate in large regions in  $\mathbb{R}^2$  without impurities. As a consequence, there the particle's potential energy is solely due to impurities outside these regions and hence to the long-distance tail of  $U$ .

The purpose of the present contribution is to summarize, unify, and generalize previous results in [BHK95, Erd98, HLW99]. Thereby we illustrate that in the presence of a magnetic field different long-distance decays of the single-impurity potential lead to a huge multiformity of Lifshits tails. Loosely speaking, there are more Landau-Lifshits tails than Lifshits tails.

The paper is organized as follows. The next section provides conditions for  $U$  to enable precise definitions of the random Landau Hamiltonian with a Poissonian potential and of its integrated density of states. The main results on Lifshits tails in magnetic fields are contained in two theorems presented in Section 3. The first theorem quotes the result for so-called super-Gaussian decay to be found in [Erd98, HLW99]. The second theorem identifies the Lifshits tails for all impurity potentials with so-called regular sub-Gaussian decay. The proof of the latter theorem is given in Section 4. Open problems related to sub-Gaussian but not regular decay are briefly discussed at the end of Section 3. For convenience, Appendix A compiles useful facts about regularly varying functions which underly our definition of regular decay. For completeness, Appendix B presents a Tauberian theorem needed in Section 4.

## 2. Basic assumptions and definitions

Throughout this paper we require the single-impurity potential  $U$  to satisfy

**Assumption 2.1.**  $U$  is positive, integrable, and locally square integrable

$$(2.1) \quad U \geq 0, \quad U \in L^1(\mathbb{R}^2) \cap L^2_{\text{loc}}(\mathbb{R}^2),$$

and strictly positive on some non-empty open set in  $\mathbb{R}^2$ .

**Remarks 2.2.** (i) Assumption 2.1 guarantees that the Poissonian potential (1.4) can be rigorously defined as a positive, measurable, ergodic random field on  $\mathbb{R}^2$  with an underlying complete probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ .

(ii) By Assumption 2.1 and [CFKS87, Theorem 1.15] the random Landau Hamiltonian  $H(V_\omega)$ , informally given by (1.3), is for  $\mathbb{P}$ -almost all  $\omega \in \Omega$  defined as a self-adjoint “magnetic” Schrödinger operator on  $L^2(\mathbb{R}^2)$ . Moreover,  $\omega \mapsto H(V_\omega)$  is measurable and ergodic with respect to (magnetic) translations.

The object of interest in this paper, the integrated density of states  $N$ , may be defined by the expectation value

$$(2.2) \quad N(E) := \int_{\Omega} d\mathbb{P}(\omega) \Theta(E - H(V_\omega))(x, x), \quad E \in \mathbb{R}.$$

Here the mapping  $\mathbb{R}^2 \times \mathbb{R}^2 \ni (x, y) \mapsto \Theta(E - H(V_\omega))(x, y)$ , denotes the continuous integral kernel of the spectral projection  $\Theta(E - H(V_\omega))$  associated with the half-line  $] -\infty, E[$ . In fact,  $N$  is the distribution function of a positive Borel measure on the real line  $\mathbb{R}$  with topological support equal to  $[\varepsilon_0, \infty[$ , the spectrum of  $H(V_\omega)$  for  $\mathbb{P}$ -almost all  $\omega$ .

**Remarks 2.3.** (i) Theorem 6.1 and Remark 6.2 (ii) in [BHL00] imply that spectral projections of  $H(V_\omega)$  indeed possess continuous integral kernels, see also [Uek94, Lemma 3.1].

(ii) Due to (magnetic) translation invariance the right-hand side of (2.2) is independent of  $x \in \mathbb{R}^2$ .

(iii) Definition (2.2) coincides with the more physical one by means of a spatial average in the macroscopic limit [Mat93, Uek94]. More precisely, by restricting  $H(V_\omega)$  to a bounded open square in  $\mathbb{R}^2$  with zero Dirichlet boundary conditions, one defines its finite-area integrated density of states to be the number of eigenvalues below  $E$  divided by the area of the square. Thanks to ergodicity of  $V_\omega$ , this quantity becomes non-random in the macroscopic limit of an unbounded square and coincides for  $\mathbb{P}$ -almost all  $\omega \in \Omega$  with  $N(E)$  except for the at most countably many discontinuity points of  $N$ .

For a general background concerning the mathematical theory of random Schrödinger operators, see [Kir89, CaLa90, PaFi92].

## 3. Lifshits tails in magnetic fields

As was already pointed out in the Introduction, the form of the Lifshits tail depends on the long-distance decay of the impurity potential  $U$ . We may classify the different decays of impurity potentials by defining the following three pairwise disjoint classes.

**Definition 3.1.** Let  $U$  be a positive measurable function on  $\mathbb{R}^2$  decaying to zero at infinity,  $\inf_{R>0} \operatorname{ess\,sup}_{|x|>R} U(x) = 0$ . Then  $U$  is said to have *super-Gaussian decay* if

$$(3.1) \quad \inf_{R>0} \operatorname{ess\,sup}_{|x|>R} \frac{\log U(x)}{|x|^2} = -\infty,$$

*Gaussian decay* if there exists a constant  $\lambda \in ]0, \infty[$  such that

$$(3.2) \quad \inf_{R>0} \operatorname{ess\,sup}_{|x|>R} \frac{\log U(x)}{|x|^2} = -\frac{1}{\lambda^2},$$

and *sub-Gaussian decay* if

$$(3.3) \quad \inf_{R>0} \operatorname{ess\,sup}_{|x|>R} \frac{\log U(x)}{|x|^2} = 0.$$

**Remarks 3.2.** (i) The condition  $U \in L^1(\mathbb{R}^2)$  is not sufficient to guarantee  $\inf_{R>0} \operatorname{ess\,sup}_{|x|>R} U(x) = 0$ .

(ii) Roughly speaking,  $U$  has super-Gaussian or sub-Gaussian decay if its upper envelope decays faster or slower than any Gaussian. The borderline case of Gaussian decay occurs when the upper envelope of  $U$  decays like a Gaussian with some definite decay length  $\lambda$ . Adopting the convention  $\log 0 := -\infty$  the class of impurity potentials with super-Gaussian decay also includes all compactly supported  $U$ . In case the long-distance decay of  $U$  is not isotropic, the above definitions always detect the slowest decay in whatever direction it may occur.

(iii) Clearly, for continuous  $U$  one may replace in the above formulae the Lebesgue-essential upper limit  $\inf_{R>0} \operatorname{ess\,sup}_{|x|>R}$  simply by  $\limsup_{|x|\rightarrow\infty}$ .

**3.1. Lifshits tails caused by super-Gaussian and Gaussian decay.** Interestingly enough, all impurity potentials  $U$  with super-Gaussian decay cause the same Lifshits tail. More precisely, one has

**Theorem 3.3.** *Let the positive impurity potential  $U$  be in  $L^2_{\text{loc}}(\mathbb{R}^2)$ , be strictly positive on some non-empty open set in  $\mathbb{R}^2$ , and have super-Gaussian decay (3.1). Then the integrated density of states has power-law fall-off to zero at  $\varepsilon_0 > 0$  in the sense that*

$$(3.4) \quad \log N(\varepsilon_0 + E) \sim \log(E^{2\pi\varrho\ell^2}) \sim -2\pi\varrho\ell^2 |\log E| \quad (E \downarrow 0).$$

**Remarks 3.4.** (i) Here and in the following we write  $F(E) \sim G(E)$  ( $E \downarrow 0$ ) and  $F(t) \sim G(t)$  ( $t \rightarrow \infty$ ) as a short-hand for *asymptotic equivalence* (at the origin and at infinity, respectively) of two real-valued functions  $F$  and  $G$  in the sense that  $\lim_{E\downarrow 0} F(E)/G(E) = 1$  and  $\lim_{t\rightarrow\infty} F(t)/G(t) = 1$ , respectively.

(ii) The Lifshits tail (3.4) exhibits a genuine quantum character because it depends, through the magnetic length  $\ell$ , on Planck's constant and the magnetic field.

(iii) The exponent  $2\pi\varrho\ell^2$  in (3.4) is just the mean number of impurities in a disk of radius  $\sqrt{2}\ell$ . Depending on whether this number is smaller or larger than one,  $N$  exhibits a root-like or true power-law fall-off (in the logarithmic sense). Both cases are sketched in Figure 1.

(iv) The theorem is proven in [HLW99] where we heavily rely on L. Erdős' result [Erd98] for compactly supported  $U$ . For its involved proof he developed a version of the “method of enlargement of obstacles” [Szn98].

In case of Gaussian decay (3.2) the question as to how the leading low-energy fall-off of  $N$  looks like is open. We only have the following result

$$(3.5) \quad -\pi\varrho(\lambda^2 + 2\ell^2) \leq \liminf_{E \downarrow 0} \frac{\log N(\varepsilon_0 + E)}{|\log E|} \leq \limsup_{E \downarrow 0} \frac{\log N(\varepsilon_0 + E)}{|\log E|} \leq -2\pi\varrho\ell^2.$$

This follows from estimating the long-distance decay of  $U$  from below and above by some super-Gaussian decay and by the definite Gaussian decay  $\lim_{|x| \rightarrow \infty} |x|^2 / \log U(x) = -\lambda^2$ , respectively. If  $U$  already has this definite Gaussian decay, the upper bound  $-2\pi\varrho\ell^2$  in (3.5) may be sharpened to  $-\pi\varrho \max\{\lambda^2, 2\ell^2\}$ . Confer [HLW99].

**3.2. Lifshits tails caused by sub-Gaussian decay.** In contrast to super-Gaussian decay, sub-Gaussian decay of the impurity potential  $U$  allows for a great variety of Lifshits tails whose details sensitively depend on the details of the decay of  $U$ . A wide class of impurity potentials able to illustrate the multiformity of Lifshits tails caused by different sub-Gaussian decay is contained in the class of functions having a definite isotropic decay in the sense of

**Definition 3.5.** A positive measurable function  $U$  on  $\mathbb{R}^2$  is said to have *regular decay*, or more specifically, a *regular  $(F, \alpha)$ -decay* if

$$(3.6) \quad \lim_{|x| \rightarrow \infty} \frac{F(1/U(x))}{|x|} = 1,$$

for some positive function  $F$ , which is regularly varying of index  $1/\alpha \in [0, \infty[$ , in symbols  $F \in R_{1/\alpha}$ , and strictly increasing towards infinity,  $F(t) \rightarrow \infty$  as  $t \rightarrow \infty$ . Here we adopt the conventions  $1/\infty := 0$  and  $1/0 := \infty$ .

**Remarks 3.6.** (i) For the definition and some useful properties of regularly varying functions, see Appendix A.

(ii) Since  $F$  is strictly increasing its inverse  $F^{-1}$  exists as a positive function on the half-line  $[\inf_{t>0} F(t), \infty[$  and is also strictly increasing. Equation (3.6) therefore requires  $U$  to be asymptotically of the form  $U(x) = 1/F^{-1}(|x|(1+o(1)))$  as  $|x| \rightarrow \infty$ , where “little oh”  $o(1)$  stands for any function decaying to zero. Moreover, since  $F \in R_{1/\alpha}$  it follows that  $F^{-1} \in R_\alpha$  (confer Appendix A). In case  $\alpha < \infty$  one therefore shows with the help of (A.6) that every  $U$  with regular decay is asymptotically of the form  $U(x) = |x|^{-\alpha} f(|x|)(1+o(1))$  as  $|x| \rightarrow \infty$ , with some  $f \in R_0$ .

(iii) Although  $F$  is required to be strictly increasing,  $U$  does not necessarily decay monotonously. However, the leading long-distance decays of its lower and upper envelope have to coincide. For example,  $U(x) = g_0 |x|^{-\alpha} [2 + (\lambda/|x|) \sin(|x|/\lambda)]$  with some constants  $g_0, \lambda, \alpha > 0$  has regular decay, but  $U(x) = g_0 |x|^{-\alpha} [2 + \sin(|x|/\lambda)]$  has not.

(iv) For a given  $U$  with regular decay the function  $F$  in Definition 3.5 is determined only up to asymptotic equivalence at infinity. This freedom may be used to choose  $F$  smooth.

(v) A function  $U$  with a regular  $(F, \alpha)$ -decay has an integrable long-distance decay if and only if  $\alpha > 2$ . It has a sub-Gaussian decay if and only if  $\lim_{t \rightarrow \infty} F(t)/\sqrt{\log t} = \infty$  or, equivalently,  $\lim_{t \rightarrow \infty} e^{-\delta t^2} F^{-1}(t) = 0$  for all  $\delta > 0$ . Consequently,  $\alpha < \infty$  implies sub-Gaussian decay.

Theorem 3.8 below deals with the class of functions having an integrable sub-Gaussian regular decay. This class is illustrated by the following four examples presented in the order of gradually slower decay.

**Examples 3.7.**

$$(i) \quad U(x) = g \exp \left[ -\frac{|x|^2}{\lambda^2 \log(|x|/\mu)} (1 + o(1)) \right] \quad \text{as } |x| \rightarrow \infty,$$

with some constants  $g, \lambda, \mu > 0$ . Here  $F(t) \sim \lambda \sqrt{(\log t) \log(\sqrt{\log t})}$  ( $t \rightarrow \infty$ ),  $\alpha = \infty$ .

$$(ii) \quad U(x) = g \exp \left[ -(|x|/\lambda)^\beta (1 + o(1)) \right] \quad \text{as } |x| \rightarrow \infty,$$

with some constants  $0 < \beta < 2$  and  $g, \lambda > 0$ . It defines the class of functions with decays called *stretched-Gaussian* in [HLW99]. Here  $F(t) \sim \lambda (\log t)^{1/\beta}$  ( $t \rightarrow \infty$ ),  $\alpha = \infty$ .

$$(iii) \quad U(x) = \frac{g_0}{|x|^\alpha} (1 + o(1)) \quad \text{as } |x| \rightarrow \infty,$$

with some constant  $g_0 > 0$  and  $\alpha > 2$ . It defines the class of functions with *integrable algebraic decay*. Here  $F(t) \sim (g_0 t)^{1/\alpha}$  ( $t \rightarrow \infty$ ).

$$(iv) \quad U(x) = \frac{g_0}{|x|^\alpha} |\log(|x|/\mu)| (1 + o(1)) \quad \text{as } |x| \rightarrow \infty,$$

with some constants  $g_0, \mu > 0$  and  $\alpha > 2$ . Here  $F(t) \sim (g_0 t)^{1/\alpha} [(\log t)/\alpha]^{1/\alpha}$  ( $t \rightarrow \infty$ ).

Remarkably, it is possible to determine rather explicitly the Lifshits tail caused by any impurity potential which shares the common decay properties of Examples 3.7.

**Theorem 3.8.** *Let the positive impurity potential  $U$  be in  $L^2_{\text{loc}}(\mathbb{R}^2)$  and have a regular  $(F, \alpha)$ -decay with  $\alpha \in [2, \infty]$ . Furthermore, let  $U$  have sub-Gaussian decay (3.3). Then the integrated density of states falls off to zero at  $\varepsilon_0 > 0$  asymptotically according to*

$$(3.7) \quad \log N(\varepsilon_0 + E) \sim -C(\alpha, \varrho) E^{2/(2-\alpha)} f^\#(E^{\alpha/(2-\alpha)}) \quad (E \downarrow 0).$$

Here  $C(\alpha, \varrho) := \frac{\alpha-2}{2} \left[ \frac{2\pi\varrho}{\alpha} \Gamma\left(\frac{\alpha-2}{\alpha}\right) \right]^{\alpha/(\alpha-2)}$ ,  $\Gamma$  denotes Euler's gamma function and the function  $f^\#$  is the de Bruijn conjugate of the function  $f : t \mapsto f(t) := [t^{-1/\alpha} F(t)]^{2\alpha/(2-\alpha)}$ .

**Remarks 3.9.** (i) For the definition of the de Bruijn conjugate and some examples of de Bruijn conjugate pairs, see Appendix A.

(ii) For the boundary case  $\alpha = \infty$  the assertion (3.7) reduces to

$$(3.8) \quad \log N(\varepsilon_0 + E) \sim -\pi\varrho f^\#(E^{-1}) \quad (E \downarrow 0),$$

where  $f^\#$  is the de Bruijn conjugate of  $f : t \mapsto [F(t)]^{-2}$ .

(iii) The Lifshits tail (3.7) neither depends on Planck's constant  $2\pi\hbar$  nor on the magnetic field  $B$  – an indication of its classical character. In fact, the asymptotic equivalence (3.7) remains valid if one substitutes for  $N(\varepsilon_0 + E)$  the *classical integrated density of states*

$$(3.9) \quad N_c(E) := \frac{m}{2\pi\hbar^2} \int_{\Omega} d\mathbb{P}(\omega) [E - V_{\omega}(0)] \Theta(E - V_{\omega}(0))$$

which is obviously independent of  $B$ ; confer [HLW99]. The magnetic field (and hence Planck's constant) will show up only in sub-leading corrections to (3.7). Even so, we stress that in case  $\alpha \in [4, \infty]$  the validity of the leading behaviour (3.7) already requires the presence of the magnetic field, see also the next remark.

(iv) As was pointed out in [BHK95], the Lifshits tail (3.7) for  $\varepsilon_0 > 0$  coincides with that for vanishing magnetic field,  $\varepsilon_0 = 0$ , at the corresponding spectral bottom in case  $U$  has an integrable algebraic decay which is slow in that  $\alpha \in ]2, 4[$ , compare Example 3.7 (iii) below with [Pas77] or [PaFi92, Corollary 9.14]. By following the lines of reasoning for vanishing magnetic field in [Pas77] or [PaFi92] and comparing the result with (3.7), it can be seen that this circumstance occurs for every  $U$  with some regular  $(F, \alpha)$ -decay provided that  $\alpha \in ]2, 4[$ .

We now return to the Examples 3.7. According to Theorem 3.8 the Lifshits tails caused by these impurity potentials turn out as follows.

**Examples 3.7 (revisited).**

- (i)  $\log N(\varepsilon_0 + E) \sim -\pi\varrho \lambda^2 |\log E| \log(|\log E|^{1/2}) \quad (E \downarrow 0),$
- (ii)  $\log N(\varepsilon_0 + E) \sim -\pi\varrho \lambda^2 |\log E|^{2/\beta} \quad (E \downarrow 0),$
- (iii)  $\log N(\varepsilon_0 + E) \sim -C(\alpha, \varrho) \left(\frac{g_0}{E}\right)^{2/(\alpha-2)} \quad (E \downarrow 0),$
- (iv)  $\log N(\varepsilon_0 + E) \sim -C(\alpha, \varrho) \left(\frac{g_0}{E}\right)^{2/(\alpha-2)} \left(\frac{|\log E|}{\alpha-2}\right)^{2/(\alpha-2)} \quad (E \downarrow 0).$

**Remark 3.10.** To our knowledge, the Lifshits tails (ii) and (iii) were first presented and proven in [HLW99] and [BHK95], respectively. In this sense, Theorem 3.8 unifies and generalizes these previous results. For the derivation of (i) and (iv) we took advantage of the relation  $f^\#(t) \sim 1/f(t)$  ( $t \rightarrow \infty$ ), valid in both cases, see Appendix A. The examples nicely illustrate the fact: the slower the long-distance decay of  $U$ , the faster the low-energy fall-off of  $N$ .

Theorem 3.8 covers many but not all impurity potentials  $U$  with sub-Gaussian (and integrable) decay, confer Remark 3.6 (iii). Unfortunately, we do not know of a general theory which determines the Lifshits tails caused by the remaining potentials. Our methods fail in general.

**3.3. Classical versus quantum Lifshits tails.** An interesting question is which long-distance decay of the impurity potential  $U$  causes a quantum Lifshits tail, in the sense that the leading fall-off of  $N(\varepsilon_0 + E)$  does not coincide with that of  $N_c(E)$  for  $E \downarrow 0$ . Theorem 3.3 and Theorem 3.8 give a partial answer. By passing from regular sub-Gaussian to super-Gaussian decay of  $U$ , the corresponding Lifshits tail changes from classical to quantum. Along such a route, Gaussian decay discriminates between classical and quantum Lifshits tailing [HLW99]. In case  $U$  has



sub-Gaussian but no regular decay, we do not know whether it causes a classical or quantum Lifshits tail. But we conjecture that the Lifshits tail of  $N$  can be universally deduced from the lower bound in the subsequently given inequalities (4.2). If this is true, genuine quantum effects in this tail should emerge for certain  $U$  having sub-Gaussian but no regular decay. For example, if  $U$  oscillates up to infinity between two functions with different regular sub-Gaussian decay (confer the second example in Remark 3.6 (iii)), the oscillation length has to compete with the magnetic length in the lower bound in (4.2) through the convolution. Clearly, the Golden-Thompson type of upper bound in (4.2) is not sharp enough to prove the above conjecture.

#### 4. Proof of Theorem 3.8

For the proof of Theorem 3.8 we follow exactly the strategy in [HLW99] and [BHK95], which in turn follow the strategy in [Pas77]. We note that the assumptions of the theorem imply Assumption 2.1. The Tauberian theorem in Appendix B (with  $\eta = \varepsilon_0$  and  $\gamma = 2/\alpha$ ) shows that the claimed low-energy fall-off of  $N$  is equivalent to the leading asymptotic fall-off

$$(4.1) \quad \lim_{t \rightarrow \infty} [F(t)]^{-2} \log \tilde{N}(t) = -\pi \varrho \Gamma\left(\frac{\alpha-2}{\alpha}\right)$$

of its shifted Laplace-Stieltjes transform  $\tilde{N}(t) := \int_0^\infty dN(\varepsilon_0 + E) e^{-tE}$ , for long “time”  $t > 0$ . To determine the long-time behaviour of  $\tilde{N}$  we use the pointwise sandwiching bounds

$$(4.2) \quad \frac{1}{2\pi\ell^2} \exp\left[-\varrho \int_{\mathbb{R}^2} d^2x \left(1 - e^{-t(|\phi_0|^2 * U)(x)}\right)\right] \\ \leq \tilde{N}(t) \leq \frac{e^{t\varepsilon_0}}{4\pi\ell^2 \sinh(t\varepsilon_0)} \exp\left[-\varrho \int_{\mathbb{R}^2} d^2x \left(1 - e^{-tU(x)}\right)\right].$$

They rely on a Jensen-Peierls and Golden-Thompson type of inequality and are proven in [HLW99] and [BHK95]. Here

$$(4.3) \quad (|\phi_0|^2 * U)(x) := \frac{1}{2\pi\ell^2} \int_{\mathbb{R}^2} d^2y e^{-|x-y|^2/2\ell^2} U(y)$$

denotes the Lebesgue convolution of  $U$  and the Gaussian probability density  $|\phi_0(x)|^2 := \exp[-|x|^2/(2\ell^2)]/(2\pi\ell^2)$ .

We proceed by deducing the long-time fall-offs of the lower and upper bound in (4.2) from the long-distance decays of  $|\phi_0|^2 * U$  and  $U$ , respectively. This is accomplished by the following

**Lemma 4.1.** *Let  $W$  be a positive integrable function on  $\mathbb{R}^2$  having a regular  $(F, \alpha)$ -decay with  $\alpha \in [2, \infty]$ . Then*

$$(4.4) \quad \lim_{t \rightarrow \infty} [F(t)]^{-2} \int_{\mathbb{R}^2} d^2x \left(1 - e^{-tW(x)}\right) = \int_{\mathbb{R}^2} d^2x \left(1 - e^{-|x|^{-\alpha}}\right) = \pi \Gamma\left(\frac{\alpha-2}{\alpha}\right).$$

Here we employ the conventions (A.7) to deal with the boundary case  $\alpha = \infty$  simultaneously.

The proof of Theorem 3.8 is then completed by showing that  $|\phi_0|^2 * U$  has the same sub-Gaussian, regular  $(F, \alpha)$ -decay as  $U$ . This is the content of

**Lemma 4.2.** *Let  $U$  be a positive integrable function on  $\mathbb{R}^2$ . If  $U$  has a sub-Gaussian, regular  $(F, \alpha)$ -decay with  $\alpha \in ]2, \infty]$ , then the convolution  $|\phi_0|^2 * U$  has the same sub-Gaussian, regular  $(F, \alpha)$ -decay.*

Basically, the proofs of both lemmata follow the proofs of Lemma 3.4 and Lemma 3.5 in [HLW99]. The details are as follows.

PROOF OF LEMMA 4.1. The substitution  $x =: F(t)\xi$  in the left integral in (4.4) yields

$$(4.5) \quad \int_{\mathbb{R}^2} d^2x \left(1 - e^{-tW(x)}\right) = [F(t)]^2 \int_{\mathbb{R}^2} d^2\xi \left(1 - e^{-tW(F(t)\xi)}\right).$$

Using the inverse  $F^{-1}$  of  $F$  and the fact that  $W$  has regular  $(F, \alpha)$ -decay (3.6), one shows that for every  $\varepsilon \in ]0, 1[$  there exists  $T_\varepsilon > 0$  such that

$$(4.6) \quad \frac{1}{F^{-1}((1+\varepsilon)F(t)|\xi|)} \leq W(F(t)\xi) \leq \frac{1}{F^{-1}((1-\varepsilon)F(t)|\xi|)}$$

for all  $t > T_\varepsilon$ . Since  $F^{-1} \in R_\alpha$ , that is,  $\lim_{t \rightarrow \infty} F^{-1}(t)/F^{-1}(t|\xi|) = |\xi|^{-\alpha}$ ,  $|\xi| > 0$ , the bounds (4.6) imply

$$(4.7) \quad \lim_{t \rightarrow \infty} tW(F(t)\xi) = |\xi|^{-\alpha}, \quad 0 < |\xi| \neq 1.$$

The claimed result now follows by interchanging limit and integration by applying the dominated-convergence theorem. In order to show that this theorem is indeed applicable we have to distinguish the cases  $\alpha < \infty$  and  $\alpha = \infty$ . In the first case, we may use the decomposition (A.6) together with Proposition A.1 to further estimate (4.6) and construct an upper bound on  $tW(F(t)\xi)$  which is independent of  $t$  and has an integrable long-distance decay. In the second case, we use (A.8) instead of (A.6) and Proposition A.1 in (4.6).  $\square$

PROOF OF LEMMA 4.2. As in the proof of Lemma 3.5 of [HLW99] we construct asymptotically coinciding upper and lower bounds on  $|\phi_0|^2 * U$ . For the upper bound we pick  $\varepsilon \in ]0, 1[$  and split the convolution integral into two integrals with domains of integration inside and outside a disk with radius  $\varepsilon|x|$  centered about the origin and estimate the two parts separately as follows

$$(4.8) \quad \int_{|y| \leq \varepsilon|x|} d^2y U(x-y) |\phi_0(y)|^2 \leq \sup_{|y| \leq \varepsilon|x|} U(x-y) \\ \leq \sup_{|y| \leq \varepsilon|x|} \frac{1}{F^{-1}((1-\varepsilon)|x-y|)} \leq \frac{1}{F^{-1}((1-\varepsilon)^2|x|)},$$

for sufficiently large  $|x|$ , since  $U$  has a regular  $(F, \alpha)$ -decay and  $F^{-1}$  is strictly increasing. Moreover, estimating the Gaussian  $|\phi_0|^2$  on the domain of integration yields

$$(4.9) \quad \int_{|y| > \varepsilon|x|} d^2y U(x-y) |\phi_0(y)|^2 \leq \frac{1}{2\pi\ell^2} e^{-\varepsilon^2|x|^2/2\ell^2} \int_{\mathbb{R}^2} d^2y U(y).$$

Since  $U$  has sub-Gaussian decay it follows that  $\lim_{t \rightarrow \infty} F^{-1}(t) e^{-\delta t^2} = 0$  for all  $\delta > 0$  such that the first term dominates the asymptotics of  $|\phi_0|^2 * U$ . Employing the facts that  $F$  is strictly increasing and in  $R_{1/\alpha}$ , we therefore arrive at

$$(4.10) \quad \liminf_{|x| \rightarrow \infty} \frac{F(1/(|\phi_0|^2 * U)(x))}{|x|} \geq \frac{(1-\varepsilon)^2}{(1+\varepsilon)^{1/\alpha}}.$$

For a lower bound we may proceed similarly

$$(4.11) \quad (|\phi_0|^2 * U)(x) \geq \int_{|y| \leq \varepsilon|x|} d^2y U(x-y) |\phi_0(y)|^2 \\ \geq \inf_{|y| \leq \varepsilon|x|} U(x-y) \int_{|z| \leq \varepsilon|x|} d^2z |\phi_0(z)|^2 \geq \frac{1-\varepsilon}{F^{-1}((1+\varepsilon)^2|x|)},$$

for sufficiently large  $|x|$ , which gives

$$(4.12) \quad \limsup_{|x| \rightarrow \infty} \frac{F(1/(|\phi_0|^2 * U)(x))}{|x|} \leq \frac{(1+\varepsilon)^2}{(1-\varepsilon)^{1/\alpha}}.$$

This completes the proof since  $\varepsilon$  may be picked arbitrarily small.  $\square$

### Appendix A. Elementary facts about “Regular Variation”

The theory of regular variation was initiated by Jovan Karamata in 1930. For the proofs of the properties quoted below and many further related results we refer to the excellent monograph [BGT89]. We recall that two functions  $F$  and  $G$  are *asymptotically equivalent*, in symbols  $F(t) \sim G(t)$  ( $t \rightarrow \infty$ ), if  $F(t)/G(t) \rightarrow 1$  as  $t \rightarrow \infty$ .

**Slow variation.** A positive, measurable function  $f$  on the positive half-line is said to be *slowly varying* (at infinity) if  $f(ct)/f(t) \rightarrow 1$  as  $t \rightarrow \infty$  holds for all  $c > 0$ , in symbols  $f \in R_0$ . Standard examples of slowly varying functions are

$$(A.1) \quad t \mapsto a_0 \prod_{j=1}^n [\log_j(t)]^{a_j} \quad \text{and} \quad t \mapsto \exp[(\log t)^a]$$

where  $a_j \in \mathbb{R}$ ,  $a \in ]0, 1[$ , and  $\log_j$  denotes the  $j$ -times iterated logarithm. In fact, for every  $f \in R_0$  there is an arbitrarily often differentiable function  $f_0$  with  $f_0(t) \sim f(t)$ . If in addition  $f$  is monotone,  $f_0$  can be chosen monotone, too. Even so, not every  $f \in R_0$  is equivalent to a monotone function. Actually, there are  $f \in R_0$  with  $\liminf_{t \rightarrow \infty} f(t) = 0$  and  $\limsup_{t \rightarrow \infty} f(t) = \infty$ . Nevertheless, the rate of growth (or decay) is bounded by any (inverse) power according to so-called Potter bounds [BGT89, Theorem 1.5.6]

**Proposition A.1.** *Let  $f \in R_0$ , then for any pair of constants  $A > 1$  and  $\delta > 0$  there exists  $T \in \mathbb{R}$ , possibly depending on  $A$  and  $\delta$ , such that*

$$(A.2) \quad f(t)/f(s) \leq A \max \left\{ (t/s)^\delta, (t/s)^{-\delta} \right\}$$

for all  $t, s \geq T$ .

**The de Bruijn conjugate.** For  $f \in R_0$  there exists  $f^\# \in R_0$ , unique up to asymptotic equivalence, such that

$$(A.3) \quad f(t) f^\#(tf(t)) \rightarrow 1, \quad f^\#(t) f(tf^\#(t)) \rightarrow 1 \quad \text{as } t \rightarrow \infty,$$

and  $f^{\#\#} \sim f$ . The function  $f^\#$  is called the *de Bruijn conjugate* of  $f$  and  $(f, f^\#)$  is referred to as a *conjugate pair*. With positive constants  $A, B, \beta > 0$  each of the following three pairs is a conjugate pair:

$$(A.4) \quad (f(At), f^\#(Bt)), \quad (Af(t), A^{-1}f^\#(t)), \quad ([f(t^\beta)]^{1/\beta}, [f^\#(t^\beta)]^{1/\beta}).$$

In some cases, the de Bruijn conjugate can be calculated explicitly, confer [BGT89, Appendix 5]. Notably, if  $f(tf(t)) \sim f(t)$  ( $t \rightarrow \infty$ ) then  $f^\#(t) \sim 1/f(t)$ . This

simple criterion applies, for instance, to the first example in (A.1) and gives in particular that  $([\log t]^\beta, [\log t]^{-\beta})$  is a conjugate pair for any real  $\beta \neq 0$ . For the de Bruijn conjugate of the second example in (A.1) there are also explicit expressions available which are somewhat complicated if  $a \geq 1/2$ .

**Regular variation.** A positive measurable function  $F$  is said to be *regularly varying* (at infinity) if  $\lim_{t \rightarrow \infty} F(ct)/F(t) \in ]0, \infty[$  for all  $c > 0$  in a set of strictly positive Lebesgue measure. Then there is  $\gamma \in \mathbb{R}$  such that

$$(A.5) \quad \lim_{t \rightarrow \infty} \frac{F(ct)}{F(t)} = c^\gamma,$$

for all  $c > 0$ . We call such a  $F$  *regularly varying of index  $\gamma$*  and write  $F \in R_\gamma$ . Every  $F \in R_\gamma$  has the form

$$(A.6) \quad F(t) = t^\gamma f(t)$$

with some  $f \in R_0$ . For  $\gamma \in \mathbb{R}$ ,  $\delta > 0$ ,  $F \in R_\gamma$ , and  $G \in R_\delta$  one has  $F(G(\cdot)) \in R_{\gamma\delta}$ . Every  $F \in R_\gamma$  with  $\gamma \neq 0$  is asymptotically equivalent to a monotone function. Its inverse belongs to  $R_{1/\gamma}$ . More explicitly, if  $F(t) \sim t^{\gamma\delta}(f(t^\delta))^\gamma$  with some  $f \in R_0$  and  $\gamma, \delta > 0$  and  $G$  is an *asymptotic inverse* of  $F$ , that is,  $G(F(t)) \sim F(G(t)) \sim t$ , then  $G(t) \sim t^{1/(\gamma\delta)}(f^\#(t^{1/\gamma}))^{1/\delta}$ .

**Rapid variation.** The boundary cases  $\gamma = \pm\infty$  in (A.5) lead to the notion of rapidly varying functions, where we adopt the conventions

$$(A.7) \quad c^\infty := \begin{cases} 0 & 0 < c < 1 \\ 1 & c = 1 \\ \infty & c > 1 \end{cases} \quad \text{and} \quad c^{-\infty} := \begin{cases} \infty & 0 < c < 1 \\ 1 & c = 1 \\ 0 & c > 1. \end{cases}$$

More precisely, a positive measurable function  $F$  is said to be *rapidly varying of index  $\pm\infty$*  if (A.5) holds with  $\gamma = \pm\infty$  for all  $c > 0$ , in symbols  $F \in R_{\pm\infty}$ . If  $F \in R_\infty$  is non-decreasing, then for any  $A < 1$  and  $\gamma \in \mathbb{R}$  there exists  $T > 0$  such that

$$(A.8) \quad \frac{F(ct)}{F(t)} \geq Ac^\gamma$$

for all  $t > T$  and  $c \geq 1$ . Moreover, if  $F \in R_0$  with  $F(t) \rightarrow \infty$  as  $t \rightarrow \infty$  and  $G$  is an asymptotic inverse of  $F$ , then  $G \in R_\infty$ .

## Appendix B. A Tauberian theorem of exponential type

**Theorem B.1.** *Let  $N$  be the distribution function of a positive Borel measure on the real line  $\mathbb{R}$ . Assume there is a constant  $\eta \in \mathbb{R}$  such that  $N(E) = 0$  for all  $E \leq \eta$ . Moreover, define the shifted Laplace-Stieltjes transform of  $N$  by*

$$(B.1) \quad \tilde{N}(t) := \int_0^\infty dN(\eta + E) e^{-tE} = e^{t\eta} \int_\eta^\infty dN(E) e^{-tE}, \quad t > 0,$$

*and suppose that  $\tilde{N}(\tau) < \infty$  for some  $\tau > 0$ . Let  $(f, f^\#)$  be a conjugated pair of slowly varying functions and  $\gamma \in [0, 1[$ . Then*

$$(B.2) \quad \log \tilde{N}(t) \sim -t^\gamma [f(t)]^{\gamma-1} \quad (t \rightarrow \infty),$$

*if and only if*

$$(B.3) \quad \log N(\eta + E) \sim -(1-\gamma) \left(\frac{\gamma}{E}\right)^{\gamma/(1-\gamma)} f^\#(E^{1/(\gamma-1)}) \quad (E \downarrow 0).$$

**Remarks B.2.** (i) For  $\gamma > 0$  the theorem is due to de Bruijn as one may check by setting  $A = 1$ ,  $B = (\gamma - 1)/\gamma$ ,  $\beta = \gamma/(\gamma - 1)$ , and  $L(E) = \gamma^{\gamma/(1-\gamma)} [f^\#(E^{1/\gamma})]^\gamma$  in [Bru59, Theorem 2], see also [BGT89, Theorem 4.12.9]. It was re-discovered in a slightly different formulation by Minlos and Povzner [MiPo67, Appendix], see also [PaFi92, Theorem 9.7].

(ii) For the boundary case  $\gamma = 0$  the assertion of the theorem reduces to

$$(B.4) \quad \lim_{t \rightarrow \infty} f(t) \log \tilde{N}(t) = -1 \quad \text{if and only if} \quad \lim_{E \downarrow 0} \frac{\log N(\eta + E)}{f^\#(1/E)} = -1.$$

The equivalence (B.4) was proven in [HLW99] only for  $f(t) = C(\log t)^{-\beta}$ ,  $C > 0$ ,  $\beta \geq 1$ . For  $f(t) = C(\log t)^{-1}$  it is a corollary of one of Karamata's early results [Kar31].

(iii) Integrating by parts in (B.1) gives

$$(B.5) \quad \tilde{N}(t) = t \int_{\eta}^{\infty} dE N(E) e^{-t(E-\eta)}, \quad t > \tau.$$

PROOF OF THEOREM B.1 FOR  $\gamma = 0$ . We will only give an outline since the proof copies exactly the strategy of [Par61]. First note that the theorem is immediate if  $f(t) \rightarrow c > 0$  as  $t \rightarrow \infty$  since  $\lim_{E \downarrow 0} N(\eta + E) = \lim_{t \rightarrow \infty} \tilde{N}(t)$ . We will therefore assume throughout the rest  $f(t) \rightarrow 0$  ( $f^\#(t) \rightarrow \infty$ ) as  $t \rightarrow \infty$  and, moreover,  $\eta = 0$  without loss of generality. Using Lemma B.3 below one shows that for every  $\varepsilon > 0$

$$(B.6) \quad \limsup_{E \downarrow 0} \frac{\log N(E)}{f^\#(1/E)} \leq -1 + \varepsilon \quad \text{and} \quad \liminf_{t \rightarrow \infty} f(t) \log \tilde{N}(t) \geq -1 - \varepsilon$$

provided that (B.2) and (B.3) holds, respectively. To complete the proof we note that Lemma B.4 below gives

$$(B.7) \quad \liminf_{E \downarrow 0} \frac{\log N(E)}{f^\#(1/E)} \geq -1 \quad \text{and} \quad \limsup_{t \rightarrow \infty} f(t) \log \tilde{N}(t) \leq -1,$$

again supposing that (B.2) and (B.3) holds, respectively.  $\square$

**Lemma B.3.** *In the setting of Theorem B.1 assume  $\gamma = \eta = 0$ . Then for every  $\varepsilon > 0$  and  $E > 0$*

$$(B.8) \quad N(E) \leq e^{\varepsilon f^\#(1/E)} \tilde{N}\left(\varepsilon \frac{f^\#(1/E)}{E}\right).$$

PROOF. Since  $N(E) \leq e^{tE} \tilde{N}(t)$  for all  $t > 0$  (compare [HLW99, Equation (A.4)]) the inequality follows by choosing  $t = \varepsilon f^\#(1/E)/E$ .  $\square$

**Lemma B.4.** *In the setting of Theorem B.1 assume  $\gamma = \eta = 0$  and that either (B.2) or (B.3) holds. If furthermore  $f(t) \rightarrow 0$  as  $t \rightarrow \infty$ , then*

$$(B.9) \quad N(E) \geq \tilde{N}\left(\frac{f^\#(1/E)}{E}\right) - 2 \exp\left(-\frac{9}{8} f^\#(1/E)\right)$$

for sufficiently small  $E > 0$ .

PROOF. We put  $t_E := f^\#(1/E)/E$  and split the domain of integration in (B.5) into the three parts  $[0, E]$ ,  $[E, 2E]$ , and  $[2E, \infty[$ . The integral over the first part is estimated according to

$$(B.10) \quad t_E \int_0^E du N(u) e^{-t_E u} \leq N(E)$$

due to monotonicity of  $N$ . We now employ Lemma B.3 or (B.3) directly to show that  $N(E) \leq \exp(-f^\#(1/E)/2)$  for sufficiently small  $E > 0$  which implies the following upper bound for the integral over the second part

$$(B.11) \quad t_E \int_E^{2E} du N(u) e^{-t_E u} \leq t_E \int_E^{2E} du \exp \left[ -t_E u \left( 1 + \frac{E}{2u} \frac{f^\#(1/u)}{f^\#(1/E)} \right) \right] \\ \leq t_E \int_E^{2E} du \exp \left( -\frac{9}{8} t_E u \right) \leq \exp \left( -\frac{9}{8} f^\#(1/E) \right).$$

Here we used Proposition A.1 with  $A = \sqrt{2}$  and  $\delta = 1$  which yields  $f^\#(1/u)/u \geq f^\#(1/2E)/(2\sqrt{2}E) \geq f^\#(1/E)/(4E)$  for sufficiently small  $E > 0$ , since  $u \in [E, 2E]$  and  $f^\# \in R_0$ . Finally,

$$(B.12) \quad t_E \int_{2E}^\infty du N(u) e^{-t_E u} \leq \exp \left( -\frac{3}{2} f^\#(1/E) \right) t_E \int_{2E}^\infty du N(u) e^{-t_E u/4} \\ \leq 4\tilde{N} \left( \frac{t_E}{4} \right) \exp \left( -\frac{3}{2} f^\#(1/E) \right) \leq \exp \left( -\frac{9}{8} f^\#(1/E) \right)$$

for sufficiently small  $E > 0$ , for which  $\tilde{N}(t_E/4) < \tilde{N}(\tau) < \infty$  and  $f^\#(1/E) \rightarrow \infty$  as  $E \downarrow 0$ .  $\square$

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### References

- [BGT89] N. H. Bingham, C. M. Goldie and J. L. Teugels, *Regular variation*, Encyclopedia of Mathematics and its Applications 27, paperback ed. with additions (Cambridge University Press, Cambridge, 1989).
- [BHL95] K. Broderix, D. Hundertmark, W. Kirsch and H. Leschke, *The fate of Lifshits tails in magnetic fields*, J. Stat. Phys. **80** (1995) 1–22.
- [BHL00] K. Broderix, D. Hundertmark and H. Leschke, *Continuity properties of Schrödinger semigroups with magnetic fields*, e-print 1998, math-ph/9808004, to appear in: Rev. Math. Phys.
- [Bru59] N. G. de Bruijn, *Pairs of slowly oscillating functions occurring in asymptotic problems concerning the Laplace transform*, Nieuw Archief voor Wiskunde (3) **VII** (1959) 20–26.
- [CaLa90] R. Carmona and J. Lacroix, *Spectral theory of random Schrödinger operators* (Birkhäuser, Boston, 1990).
- [CFKS87] H. L. Cycon, R. G. Froese, W. Kirsch and B. Simon, *Schrödinger operators* (Springer, Berlin, 1987).
- [Erd98] L. Erdős, *Lifshits tail in a magnetic field: the nonclassical regime*, Probab. Theory Relat. Fields **112** (1998) 321–371.
- [Foc28] V. Fock, *Bemerkung zur Quantelung des harmonischen Oszillators im Magnetfeld*, Z. Physik **47** (1928) 446–448 (in German).

- [HLW99] T. Hupfer, H. Leschke and S. Warzel, *Poissonian obstacles with Gaussian walls discriminate between classical and quantum Lifshits tailing in magnetic fields*, to appear in: J. Stat. Phys. **97** (1999).
- [Kar31] J. Karamata, *Neuer Beweis und Verallgemeinerung der Tauberschen Sätze, welche die Laplacesche und Stieltjessche Transformation betreffen*, J. Reine Angew. Math. **164** (1931) 27–39 (in German).
- [Kir89] W. Kirsch, *Random Schrödinger operators: a course*, in: H. Holden and A. Jensen (eds.), *Schrödinger operators*, Lecture Notes in Physics **345** (Springer, Berlin, 1989) pp. 264–370.
- [Lan30] L. Landau, *Diamagnetismus der Metalle*, Z. Physik **64** (1930) 629–637 (in German).
- [Lif64] I. M. Lifshitz, *The energy spectrum of disordered systems*, Adv. Phys. **13** (1964) 483–536. *Energy spectrum structure and quantum states of disordered condensed systems*, Sov. Phys. Usp. **7** (1965) 549–573. Russian original: Usp. Fiz. Nauk. **83** (1964) 617–663.
- [LGP88] I. M. Lifshits, S. A. Gredeskul and L. A. Pastur, *Introduction to the theory of disordered systems* (Wiley, New York, 1988). Russian original: Nauka, Moscow, 1982.
- [Mat93] H. Matsumoto, *On the integrated density of states for the Schrödinger operators with certain electromagnetic potentials*, J. Math. Soc. Japan **45** (1993) 197–214.
- [MiPo67] R. A. Minlos and A. Ja. Povzner, *Thermodynamic limit for entropy*, Trans. Moscow Mat. Soc. **17** (1967) 269–300. Russian original: Trudy Moskow Mat. Obsc. **17** (1967) 243–272.
- [Par61] S. Paramešwaran, *Partition functions whose logarithms are slowly oscillating*, Trans. Amer. Math. Soc. **100** (1961) 217–240.
- [Pas77] L. A. Pastur, *Behavior of some Wiener integrals as  $t \rightarrow \infty$  and the density of states of Schrödinger equations with random potential*, Theor. Math. Phys. **32** (1977) 615–620. Russian original: Teor. Mat. Fiz. **32** (1977) 88–95.
- [PaFi92] L. Pastur and A. Figotin, *Spectra of random and almost-periodic operators* (Springer, Berlin, 1992).
- [Szn98] A.-S. Sznitman, *Brownian motion, obstacles and random media* (Springer, Berlin, 1998).
- [Uek94] N. Ueki, *On spectra of random Schrödinger operators with magnetic fields*, Osaka J. Math. **31** (1994) 177–187.
- [Wey12] H. Weyl, *Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung)*, Math. Ann. **71** (1912) 441–479 (in German).

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